Quasiattractor dynamics of $\lambda\phi^4$-inflation

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At high e-foldings of expansion, the inflation with the quartic potential exhibits the parametric attractor governed by the slowly running Hubble rate. This quasiattractor simplifies the analysis of predictions for the inhomogeneity generated by the quantum fluctuations of inflaton. The method reveals the connection of inflation e-folding with general parameters of preheating regime in various scenarios and observational data.

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I. INTRODUCTION

The homogeneous component of inflationary dynamics in the simplest case of single field [1, 2, 3, 4] (see review in [5]) is usually described in terms of slow-roll approximation in the evolution of scalar inflaton ϕ, when one neglects both its kinetic energy $\frac{1}{2}\dot\phi^2$ and acceleration $\ddot\phi$ in the field equations

$$
\begin{aligned}
H^2 &= \frac{8\pi}{3\,m_{\mathtt{Pl}}^2}\left\{\frac{1}{2}\,\dot\phi^2 + V(\phi)\right\},\\
\ddot\phi &= -3H\dot\phi - \frac{\partial V}{\partial\phi},
\end{aligned}
\tag{1}
$$

derived from the Hilbert–Einstein action of Friedmann–Robertson–Walker metric

$$
\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\boldsymbol{r}^2, \tag{2}
$$

with the scale factor of expansion $a(t)$ ordinary defining the Hubble rate $H = \dot a/a$ in terms of its derivative with respect to time t as denoted by over-dot $\dot\diamond \equiv \mathrm{d}\diamond/\mathrm{d}t$. In (1) the Planck mass is usually introduced through the Newton constant G by the relation $m_{\mathtt{Pl}}^2 = 1/G$, while V is the inflaton potential. The slow-rolling implies

$$
\begin{aligned}
H^2 &\approx \frac{8\pi}{3\,m_{\mathtt{Pl}}^2}\,V,\\
\dot\phi &\approx -\frac{1}{3H}\,\frac{\partial V}{\partial\phi}.
\end{aligned}
\tag{3}
$$

For the consistency of such the approximation we have to estimate the ratios

$$
\frac{\dot\phi^2}{2V} \approx \frac{1}{3}\,\frac{m_{\mathtt{Pl}}^2}{16\pi}\left(\frac{\partial\ln V}{\partial\phi}\right)^2 \equiv \frac{1}{3}\,\epsilon \ll 1, \tag{4}
$$

and

$$
\frac{\ddot\phi}{3H\dot\phi} \approx -\frac{1}{9H^2}\left\{\frac{\partial^2 V}{\partial\phi^2} - \frac{4\pi}{3m_{\mathtt{Pl}}^2}\,\frac{1}{H^2}\left(\frac{\partial V}{\partial\phi}\right)^2\right\}. \tag{5}
$$

Then, introducing parameter η by

$$
\eta = \frac{m_{\mathtt{Pl}}^2}{8\pi}\,\frac{1}{V}\,\frac{\partial^2 V}{\partial\phi^2}, \tag{6}
$$

and using (3), we get

$$
\frac{\ddot\phi}{3H\dot\phi} \approx -\frac{1}{3}\{\eta - \epsilon\}. \tag{7}
$$

Therefore, the slow-roll approximation is substantiated at

$$
\epsilon \ll 1, \quad |\eta| \ll 1. \tag{8}
$$

Constraints (8) are usually satisfied for positive power potentials, for instance, at appropriately chosen parameters and fields $\phi \gtrsim m_{\mathtt{Pl}}$, which guarantee the inflation of Universe with slowly changing Hubble rate with a huge total e-folding of scale factor $N_{\mathrm{tot.}} = \ln a_{\mathrm{end}}/a_{\mathrm{init.}}$, where subscripts stand for the moments of ending and initiating the inflation, correspondingly.

Note, that the above speculations have not involved any initial data on the field evolution. This can be meaningful, if only the evolution possesses properties of attractors, which allow a stable dynamical behavior, that soon forgets about a start.

The quasiattractor for the inflationary dynamics with the quadratic potential $V \sim m^2\phi^2$ was found in [6], which modernized the consideration of dependence on the initial conditions in the homogeneous case as was done in pioneering papers of [7, 8]. So, the phase space variables of dynamical system rapidly tend to a stable point, which position depends on the parameter defined by H, while the parameter itself slowly evolves during the most amount of e-folding in the inflation and it begins significantly to change to the end of inflation at a short increment of $N_{\mathrm{tot.}}$, only. Thus, at high total e-folding of inflation governed by an appropriate large starting value of Hubble rate, the inflation due to the quadratic potential is well described by the quasiattractor behavior. Similar result was obtained for the quartic potential in [9] for the homogeneous case, and further attempts to generalize the analysis to damping of initial inhomogeneities were formulated.

The analysis of dynamical system for the homogeneous inflaton and baryotropic matter was done in [10], wherein some general features of evolution were derived for an

arbitrary potential of inflaton, and several explicit examples were studied.

Another aspect of considering the initial conditions concerns for inhomogeneities perturbing both the inflaton and metric. The analysis of inhomogeneity in the initial values of scalar field beyond the perturbation theory as well as relevant references can be found in [11], wherein the authors formulated conditions constraining the parameters of initial state, when the inflation takes place.

In the present paper we generalize the approach of [6] to the case of quartic potential $V \sim \lambda\phi^4$, that modernizes the consideration in [9]. The main goal of our study is to show the effectiveness and elegance of quasiattractor method to the analysis of inflation dynamics and its connection with the preheating regime in the case of quartic potential in order to draw the conclusion on the consistency with the observational data versus general conditions of preheating. The dynamical system relevant to the evolution of homogeneous Universe is considered in Section II. We find the stable quasiattractor for the quartic potential, too. The inflation parameters are investigated in Section III, wherein they are confronted with both the experimental data and predictions of slow-roll approximation. Our results are summarized and discussed in Conclusion.

II. DYNAMICAL SYSTEM

The action of inflaton is of the form

$$S = \int \mathrm{d}^4 x\sqrt{-g}\left\{\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right\}, \tag{9}$$

with the quartic potential

$$V(\phi) = \frac{\lambda}{4}\,\phi^4. \tag{10}$$

In the Friedmann–Robertson–Walker metric it leads to the evolution

$$\begin{aligned} \ddot{\phi} &= -3H\dot{\phi} - \lambda\phi^3, \\ \dot{H} &= -\frac{1}{2}\,\kappa^2\dot{\phi}^2, \end{aligned} \tag{11}$$

under the Friedmann equation

$$H^2 = \frac{2}{3}\,\kappa^2\left\{\frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}\,\phi^4\right\}, \tag{12}$$

at $\kappa^2 = 8\pi G$.

Following the method of [6], let us introduce the phase space variables

$$x = \frac{\kappa}{\sqrt{6}}\,\frac{\dot{\phi}}{H}, \qquad y = \sqrt[4]{\frac{\lambda}{12}}\,\frac{\kappa\,\phi}{\sqrt{\kappa H}}, \tag{13}$$

related due to (12) by the constraint

$$x^2 + y^4 = 1. \tag{14}$$

The analysis of dynamical system is simplified, if we introduce the driving parameter

$$z = \frac{\sqrt[4]{3\lambda}}{\sqrt{\kappa H}}. \tag{15}$$

Then, denoting the derivative with respect to the amount of e-folding by prime $\diamond' \equiv \mathrm{d}\diamond/\mathrm{d}N$ at $N = \ln a - \ln a_{\text{init.}}$, we get

$$\begin{aligned} x' &= 3x^3 - 3x - 2y^3 z, \\ y' &= \frac{3}{2}\,x^2 y + xz, \\ z' &= \frac{3}{2}\,x^2 z. \end{aligned} \tag{16}$$

It is an easy task to confirm, that the system of (16) conserves the Friedmann constraint in the form of (14).

It is important to stress, that z is quasi-constant, if x is fixed at a critical point $x_c \ll 1$, hence,

$$z \approx z_{\text{init.}} e^{\frac{3}{2}x_c^2 N} \approx z_{\text{init.}}\left(1 + \frac{3}{2}\,x_c^2 N + \ldots\right), \tag{17}$$

so that the monotonic growth of z is slow in comparison with the linear increase of e-folding N due to a small slope proportional to $x_c^2 \to 0$, indeed, and one can put $z \approx z_{\text{init.}}$ until $\frac{3}{2}\,x_c^2 N \ll 1$, i.e. at rather large intervals of N. We will see later, in fact, that the stable critical point x_c^2 scales as $x_c^2 \approx \beta/(N_{\text{tot.}} - N)$ at $N_{\text{tot.}} - N \gg 1$ and $\beta = \frac{1}{3}$, so that the growth takes the form

$$z \approx z_{\text{init.}}\left(\frac{N_{\text{tot.}}}{N_{\text{tot.}} - N}\right)^{\frac{3}{2}\beta}. \tag{18}$$

Therefore, at $N \ll N_{\text{tot.}}$ the increase is quite linear, but the coefficient of $z_{\text{init.}}$ is actually suppressed $z_{\text{init.}} \ll 1$ by the conditions of attractor, as we will consistently see later.

Thus, we can treat the phase-space evolution in the $\{x, y\}$ plane as the autonomous dynamical system with external parameter z, when the driftage of z gives rise to the sub-leading correction.

A. Critical points

Solving $x' = y' = 0$ at $x \neq 0$ and $y \neq 0$, we get the relation[1]

$$3xy = -2z, \tag{19}$$

[1] The critical point $x = y = 0$ is off interest, since it does not satisfy constraint (14).

while

$$f(x^2) \equiv x^6 - x^4 + \left(\frac{2}{3}\,z\right)^4 = 0. \tag{20}$$

The cubic polynomial $f(u)$ in (20) produces two extremal points, since its derivative is equal to

$$\frac{\mathrm{d}f}{\mathrm{d}u} = u(3u-2),$$

hence, $u_1 = 0$, $u_2 = \frac{2}{3}$. Therefore, f has got positive roots, if only

$$f(u_2) < 0,$$

that yields

$$z^4 < \frac{3}{4}. \tag{21}$$

Constraint (21) guarantees the existence of critical points $x_c^2 > 0$.

Substituting

$$z^4 = \frac{3}{4}\sin^2\chi, \quad \text{at} \quad 0 < \chi \leqslant \frac{\pi}{2}, \tag{22}$$

allows us to write the real solutions of (20) in the following form

$$x_\ell^2 = \frac{1}{3}\left\{1 + 2\cos\left(\frac{2}{3}\,\chi + \frac{2}{3}\,\pi\ell\right)\right\}, \quad \ell = \{0, \pm 1\}. \tag{23}$$

At $\ell = +1$ quantity $x_+^2 \leqslant 0$, which is not a physical point, at all, while at $\ell = -1$ and $\ell = 0$

$$0 < x_-^2 \leqslant \frac{2}{3}, \qquad \frac{2}{3} \leqslant x_0^2 < 1, \tag{24}$$

correspondingly. Therefore, the case of $\ell = 0$ is covered by the case of $\ell = -1$, if we expand the interval of χ to

$$0 < \chi < \pi, \tag{25}$$

that allows us to put the critical point[2] to $x_c = x_-$.

Due to (19), we can write down

$$f(x^2) = x^4(x^2 + y^4 - 1),$$

so that the critical points satisfy the Friedmann constraint of (14).

[2] One could change the convention by setting $-\pi < \chi < 0$ and $x_c = x_0$, but we prefer for (25), when the case of interest with $x_c^2 \ll 1$ is spectacularly reached at $\chi \to 0$.

B. Stability analysis

At $x = x_c + \bar{x}$ and $y = y_c + \bar{y}$, the evolution linear in $\{\bar{x}, \bar{y}\}$ reads off as

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} 9x_c^2 - 3 & -6y_c^2 z \\ -z & \frac{3}{2}x_c^2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \tag{26}$$

wherein we have used (19).

The linearized Friedmann constraint in (14) gives

$$x_c\bar{x} = -2y_c^3\bar{y}, \tag{27}$$

so that (26) is reduced to the single equation

$$\bar{y}' = \frac{9}{2}\left(x_c^2 - \frac{2}{3}\right)\bar{y}. \tag{28}$$

Therefore, at

$$x_c^2 < \frac{2}{3}, \tag{29}$$

the deviations $\{\bar{x}, \bar{y}\}$ will be damped, while at $x_c \to 0$, the fluctuations will decline as $1/N^3$.

The same fact can be found in a general manner. So, the eigenvalues of matrix in (26) are the following:

$$\lambda_+ = 6x_c^2, \qquad \lambda_- = \frac{9}{2}\left(x_c^2 - \frac{2}{3}\right), \tag{30}$$

while the corresponding eigenvectors are given by

$$v_+ = \begin{pmatrix} -\frac{9}{2}\,x_c^2 \\ z \end{pmatrix}, \qquad v_- = \begin{pmatrix} 3y_c^4 \\ z \end{pmatrix}. \tag{31}$$

However, the condition of (27) can be rewritten as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 3y_c^4 \\ z \end{pmatrix}\frac{\bar{y}}{z} = v_-\,\frac{\bar{y}}{z}. \tag{32}$$

Therefore, the eigenvector v_+ is irrelevant to the consideration, since it drives out of constraint (27), while v_- is consistent with (27), and the critical point is stable at $\lambda_- < 0$, i.e. under (29).

Thus, we have established the quasiattractor for the quartic potential of inflaton. The stability of critical point is controlled by smallness of z, that means large enough value of initial Hubble rate.

III. INFLATION PARAMETERS

Staring the evolution of Universe filled by the inflaton from a high initial Hubble rate, i.e. at $z_{\text{init.}} \to 0$, but $H^2_{\text{init.}} < m^2_{\text{Pl}}$, causes a rapid entering the quasiattractor regime, when the expansion is described by the slow drift of Hubble rate. It means that a small domain of space is inflationary increased to a huge size, and the inflaton can be approximated by an almost homogeneous field versus the spatial coordinates, while its value and time derivative evolve in accordance with the drift of critical points.

A. Homogeneous limit

The quasiattractor is the reason for the cosmological evolution forgets its primary origin to the leading homogeneous approximation. Nevertheless, one could consider some properties of attractor dynamics close to the end of inflation, since it further enters the properties of observed inhomogeneity.

1. Acceleration conditions

The accelerated expansion takes place at $\ddot{a}>0$, hence,

$$\frac{\ddot{a}}{a}=\dot{H}+H^2>0,$$

that is reduced to

$$\frac{\dot{H}}{H^2}=-\frac{\kappa^2\dot{\phi}^2}{2H^2}=-3x^2>-1, \tag{33}$$

wherein we have used (11). Therefore, the Universe follows the inflation at

$$x_c^2<x_{c,\mathrm{end}}^2\equiv\frac{1}{3}, \tag{34}$$

i.e. in the region, where the quasiattractor is in action, indeed.

The accelerated regime ends at

$$\begin{aligned} y_{c,\mathrm{end}}^4&=1-x_{c,\mathrm{end}}^2=\frac{2}{3},\\ z_{c,\mathrm{end}}^4&=\left(\frac{3}{2}\,x_{c,\mathrm{end}}y_{c,\mathrm{end}}\right)^4=\frac{3}{8}, \end{aligned} \tag{35}$$

equivalent to $\chi_{\mathrm{end}}=\frac{\pi}{4}$ and

$$\phi_{\mathrm{end}}^2=\frac{8}{\kappa^2},\quad H_{\mathrm{end}}^2=\frac{8\lambda}{\kappa^2},\quad \dot{\phi}_{\mathrm{end}}^2=\frac{48\lambda}{\kappa^4}. \tag{36}$$

The inflation condition of (34) is supplemented by accompaniments

$$y_c^4>\frac{2}{3},\qquad z^4<\frac{3}{8}. \tag{37}$$

Note, that to the end of inflation the field takes the value of the order of Planck mass independently of coupling constant λ, while the corresponding squares of Hubble rate and kinetic energy are scaled linearly in the coupling constant, and they are actually suppressed to the appropriate Planck mass powers at $\lambda\ll 1$.

2. Amount of e-folding

Once the attractor is in action, the total amount of e-folding during the inflation can be obtained by integration of z-component in the system of (16) at $x=x_c$, i.e.

$$N_{\mathrm{tot.}}=\frac{2}{3}\int\limits_{z_{\mathrm{init.}}}^{z_{\mathrm{end}}}\frac{\mathrm{d}z}{x_c^2 z}. \tag{38}$$

In the limit of $z_{\mathrm{init.}}\to 0$, we deduce (38) by transformation to the integration versus χ, so that

$$N_{\mathrm{tot.}}=\int\limits_{\chi_{\mathrm{init.}}}^{\pi/4}\frac{\cot\chi\,\mathrm{d}\chi}{1+2\cos\left(\frac{2}{3}\,\chi-\frac{2}{3}\,\pi\right)}, \tag{39}$$

approximated by

$$N_{\mathrm{tot.}}^{(0)}\approx\frac{\sqrt{3}}{2}\int\limits_{\chi_{\mathrm{init.}}}^{\pi/4}\frac{\mathrm{d}\chi}{\chi^2}, \tag{40}$$

to the leading order at $\chi_{\mathrm{init.}}\to 0$. The consequent expansion in (39) gives

$$\begin{aligned} N_{\mathrm{tot.}}^{(0)}&=\frac{\sqrt{3}}{2}\left\{\frac{1}{\chi_{\mathrm{init.}}}-\frac{4}{\pi}\right\},\\ N_{\mathrm{tot.}}^{(1)}&=\frac{\sqrt{3}}{2}\left\{\frac{1}{\chi_{\mathrm{init.}}}-\frac{4}{\pi}\right\}+\frac{1}{6}\ln\frac{4\,\chi_{\mathrm{init.}}}{\pi}. \end{aligned} \tag{41}$$

The inversion of (41) results, for instance, in

$$\chi_{\mathrm{init.}}^{(0)}=\frac{\pi\sqrt{3}}{2\pi N_{\mathrm{tot.}}^{(0)}+4\sqrt{3}}. \tag{42}$$

To the leading order at $N_{\mathrm{tot.}}\gg 1$ we deduce the initial data

$$z_{\mathrm{init.}}^2\approx\frac{3}{4}\,\frac{1}{N_{\mathrm{tot.}}},\qquad x_{c,\mathrm{init.}}^2\approx\frac{1}{3}\,\frac{1}{N_{\mathrm{tot.}}}, \tag{43}$$

while $y_{c,\mathrm{init.}}^4=1-x_{c,\mathrm{init.}}^2$. Note, that the initial data for the field and its velocity $\{y,x\}$ mean their values *just after entering the quasiattractor*, while the actual data are irrelevant, since the quasiattractor rapidly adjust the field and its velocity to (43) in accordance to the initial Hubble rate, i.e. $z_{\mathrm{init.}}$.

B. Inhomogeneity

Quantum fluctuations of inflaton near its homogeneous classical value result in the primordial spatial perturbations of energy density, involving the scalar and tensor components of spectrum versus the wave vector k at the moment, when the fluctuation comes back from the horizon, i.e. at $k=a(t)\,H$ as following:

$$\begin{aligned} \mathcal{P}_{\mathrm{S}}(k)&=\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\phi}}\right)^2,\\ \mathcal{P}_{\mathrm{T}}(k)&=8\kappa^2\left(\frac{H}{2\pi}\right)^2. \end{aligned} \tag{44}$$

The spectra can be accurately evaluated in terms of quasiattractor dynamics by

$$\begin{aligned}\mathcal{P}_\mathrm{S}(k) &= \frac{\lambda}{8\pi^2}\,\frac{1}{z^4 x_\mathrm{c}^2},\\ \mathcal{P}_\mathrm{T}(k) &= \frac{6\lambda}{\pi^2}\,\frac{1}{z^4},\end{aligned}\tag{45}$$

where z corresponds to the Hubble rate H, and it is expressed by amount of e-folding N between the moments of horizon exit and inflation end, as that can be found in the right analog of (43) in the leading order at $N \gg 1$. So, we redefine $N = \ln a_\mathrm{end} - \ln a$. Then,

$$\begin{aligned}\mathcal{P}_\mathrm{S}(k) &\approx \frac{2\lambda}{3\pi^2}\,N^3,\\ \mathcal{P}_\mathrm{T}(k) &\approx \frac{32\lambda}{3\pi^2}\,N^2,\end{aligned}\tag{46}$$

which are consistent with the slow-roll approximation for the inflation with the quartic potential. The sub-leading corrections are given by the substitution following from (42),

$$\frac{1}{N} \mapsto \frac{2\pi}{2\pi N + 4\sqrt{3}}.\tag{47}$$

The ratio

$$r = \frac{\mathcal{P}_\mathrm{T}}{\mathcal{P}_\mathrm{S}} = 48\,x_\mathrm{c}^2 \ll 1,\tag{48}$$

determines the relative contribution of tensor spectrum. In the leading order

$$r \approx \frac{16}{N},\tag{49}$$

in accordance with the slow-roll approximation, again.

The spectral index is defined by

$$n_\mathrm{S} - 1 \equiv \frac{\mathrm{d}\ln\mathcal{P}_\mathrm{S}}{\mathrm{d}\ln k}.\tag{50}$$

It can be calculated under the condition[3]

$$\ln\frac{k}{k_\mathrm{end}} = -N - 2\ln\frac{z}{z_\mathrm{end}},\tag{51}$$

so that

$$n_\mathrm{S} - 1 = -6x_\mathrm{c}^2\,\frac{3-4x_\mathrm{c}^2}{(1-3x_\mathrm{c}^2)(2-3x_\mathrm{c}^2)},\tag{52}$$

that is reduced to the slow-roll result in the leading approximation

$$n_\mathrm{S} - 1 \approx -9x_\mathrm{c}^2 \approx -\frac{3}{N}.\tag{53}$$

[3] Note, that e-folding counts backward, from the end of inflation to the moment t, hence, the derivatives in (16) change sign.

To the same approximation

$$n_\mathrm{S} - 1 \approx -\frac{3}{16}\,r.\tag{54}$$

The running of spectral index is given by

$$\alpha = \frac{\mathrm{d}n_\mathrm{S}}{\mathrm{d}\ln k} = -36x_\mathrm{c}^4\,\frac{(6-16x_\mathrm{c}^2+9x_\mathrm{c}^4)(1-x_\mathrm{c}^2)}{(2-3x_\mathrm{c}^2)^3(1-3x_\mathrm{c}^2)^3},\tag{55}$$

approximately equal to

$$\alpha \approx -27x_\mathrm{c}^4 \approx -\frac{3}{N^2}.\tag{56}$$

As it was noted in [6], the evaluation of α in the framework of slow-roll approximation is quite complicated, while the quasiattractor dynamics allows us to get it straightforwardly by means of taking the derivative. Though one could use (51) to the leading order, when $\mathrm{d}k/\mathrm{d}N \approx -1$, that straightforwardly yields the spectral index as well as its slope derived above.

Thus, the quasiattractor behavior provides us with the direct way of calculating the properties of primary inhomogeneity in terms of two parameters: the coupling constant λ and number of e-folding N relative to the observational scale k.

1. *Constraining a maximal N*

Following the method elaborated in [12], let us restrict the scale $k = aH$ relative to the observation of inhomogeneity, as it comes from the both known and suggested history of Universe evolution by considering the ratio

$$\frac{k}{a_0 H_0} = \frac{aH}{a_0 H_0},$$

where H_0 is the present-date Hubble rate

$$H_0 = 1.75\cdot 10^{-61}\,h\,m_\mathtt{Pl}, \qquad h \simeq 0.7,\tag{57}$$

while a_0 is the present-date scale factor, that can be set to $a_0 \equiv 1$ by definition of length unit. We decompose the evolution in several consequent stages: the inflation, reheating, radiation dominance up to the epoch of equality with nonrelativistic matter, matter dominance, that give

$$\begin{aligned}\frac{a}{a_0} &= \frac{a}{a_\mathrm{end}}\,\frac{a_\mathrm{end}}{a_\mathrm{reh.}}\,\frac{a_\mathrm{reh.}}{a_\mathrm{eq.}}\,\frac{a_\mathrm{eq.}}{a_0},\\ \frac{H}{H_0} &= \frac{H}{H_\mathrm{end}}\,\frac{H_\mathrm{end}}{H_\mathrm{eq.}}\,\frac{H_\mathrm{eq.}}{H_0}.\end{aligned}\tag{58}$$

The data yield

$$H_\mathrm{eq.} \simeq 5.25\cdot 10^6\,h^3\Omega_\mathrm{M}^2 H_0, \qquad \frac{a_\mathrm{eq.}H_\mathrm{eq.}}{a_0 H_0} \simeq 219\,h\,\Omega_\mathrm{M},\tag{59}$$

where Ω_M is the fraction of matter energy. The quasiattractor gives

$$\frac{H^2}{H_\mathrm{end}^2} = \frac{z_\mathrm{end}^4}{z^4} = \frac{2}{3}\,N^2.\tag{60}$$

The scale ratios are related with the ratios of Hubble rates in two subsequent stages "b" $\to$ "c" by means of

$$H_b^2 = H_c^2\left(\frac{a_c}{a_b}\right)^{3(1+w_c)}, \tag{61}$$

where w_c is the state parameter at the stage "c", determined by the ratio of pressure to the energy density. For the radiation stage we set $w_{\rm eq.}=\frac{1}{3}$.

Then, we arrive to

$$\begin{aligned} N = &-\ln\frac{k}{a_0H_0}+\ln 219\,\Omega_{\rm M}h+\frac{1}{2}\ln\frac{2}{3}+\ln N \\ &+\left(\frac{1}{2}-\frac{1}{3(1+w_{\rm reh.})}\right)\ln\frac{H^2_{\rm end}}{m^2_{\tt Pl}} \\ &-\left(\frac{1}{4}-\frac{1}{3(1+w_{\rm reh.})}\right)\ln\frac{H^2_{\rm reh.}}{m^2_{\tt Pl}}-\frac{1}{4}\ln\frac{H^2_{\rm eq.}}{m^2_{\tt Pl}}. \end{aligned} \tag{62}$$

Let us parameterize the Hubble rate at the end of reheating by a scale $\mu_{\rm reh.}$, so that

$$H^2_{\rm reh.} = \frac{8\pi}{3\,m^2_{\tt Pl}}\,\mu^4_{\rm reh.}, \tag{63}$$

while

$$\frac{H^2_{\rm end}}{m^2_{\tt Pl}} = \frac{3\pi}{2N^3}\,\mathcal{P}_{\rm S}. \tag{64}$$

Therefore,

$$\begin{aligned} N = &-\ln\frac{k}{a_0H_0}+\ln 219\,\Omega_{\rm M}h+\frac{1}{2}\ln\frac{2}{3} \\ &-\left(\frac{1}{2}-\frac{1}{1+w_{\rm reh.}}\right)\ln N-\frac{1}{4}\ln\frac{H^2_{\rm eq.}}{m^2_{\tt Pl}} \\ &+\left(\frac{1}{2}-\frac{1}{3(1+w_{\rm reh.})}\right)\ln\frac{3\pi}{2}\,\mathcal{P}_{\rm S} \\ &-\left(\frac{1}{4}-\frac{1}{3(1+w_{\rm reh.})}\right)\ln\frac{8\pi}{3}\,\frac{\mu^4_{\rm reh.}}{m^4_{\tt Pl}}. \end{aligned} \tag{65}$$

Note, that (65) is independent of current matter density $\Omega_{\rm M}$.

Astronomical observations are naturally restricted by the distances less than the event horizon given by the inverse Hubble rate of present day H_0. Therefore, the maximal value of e-folding N_0 is reached at the wave number $k\mapsto k_0=a_0H_0$, when (65) contains two parameters determined by the mechanism of reheating. In addition, experimental data give the spectral density [14]

$$\mathcal{P}_{\rm S}\simeq 2.5\cdot 10^{-9} \tag{66}$$

at the scale $k\simeq 0.05-0.002$ Mpc^{-1} greater than k_0, nevertheless, this value of $\mathcal{P}_{\rm S}$ rather slowly evolves with k.

The dependence of N_0 versus the scale of reheating $\mu_{\rm reh.}$ and state parameter $w_{\rm reh.}$ is shown in Fig. 1 for generic cases of stiff matter $w=1$, radiation $w=\frac{1}{3}$ and dust $w=0$. It is spectacular, that, first, the dependence versus the scale is well approximated by logarithmic law, second, all these reheating regimes converge to a single point at $\bar\mu_{\rm reh.}\simeq 0.33\cdot 10^{16}$ GeV, and third, the radiation regime gives the maximal amount of e-folding[4] equal to $N_{0,\rm rad.}\approx 64$ independent of the scale characterizing the reheating stage. The reheating scale $\bar\mu_{\rm reh.}$ for the intersection of curves with different values of state parameter $w_{\rm reh.}$ in Fig. 1 is given by the condition of nullifying the coefficient in front of factor $1/(1+w_{\rm reh.})$ in (65), so that

$$\bar\mu^4_{\rm reh.} = \frac{9}{16}\,\frac{\mathcal{P}_{\rm S}}{N^3_{0,\rm rad.}}\,m^4_{\tt Pl}. \tag{67}$$

Our numerical calculations exhibit the oscillations of the field around the minimum of quartic potential with $w_{\rm reh.}\approx 1$. However, this value can be modified by the both corrections to the potential near the minimum and inflaton interaction with ordinary fields.

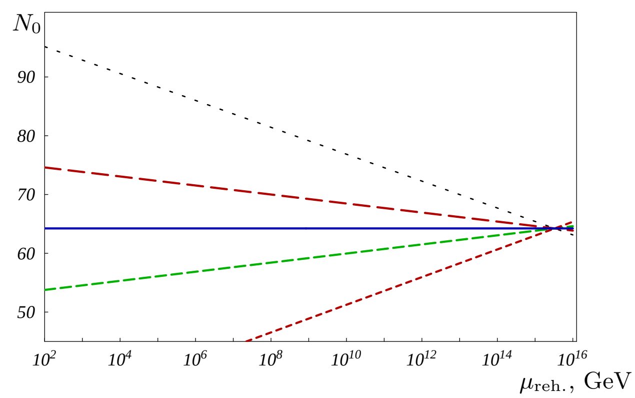

FIG. 1: The maximal number of e-folding N_0 versus the reheating scale in different regimes of reheating state parameter $w_{\rm reh.}=1$ (long-dashed line), $w_{\rm reh.}=\frac{1}{3}$ (solid line), $w_{\rm reh.}=0$ (dashed line), $w_{\rm reh.}\to\infty$ (dotted line), and $w_{\rm reh.}=-\frac{1}{3}$ (short-dashed line).

We have also shown two marginal cases. The first case corresponds to $w_{\rm reh.}=-\frac{1}{3}$ and it could appear, if we suggest the inflaton oscillations near the potential minimum neglecting the gravitational damping during the period. Then, the virial theorem gives averages of kinetic and potential energies as $\langle x^2\rangle=\frac{1}{2}\langle y^4\rangle$, that reproduce the mentioned state parameter.

The second case corresponds to a quite instantaneous falling of Hubble rate to its reheating scale, so that $w_{\rm reh.}\to\infty$. Such the regime could happen due to a specific tachyonic preheating [13] with a rapid decay of classical inflaton to quanta, for instance. However, the

[4] The value of $N_{0,\rm rad.}$ is obtained by numerical solving (65) at $w=\frac{1}{3}$.

jump of Hubble rate is the indication of energy jump, that is unrealistic, at all. Nevertheless, the situation with $w > 1$ takes place, if the potential minimum is shifted to a negative value $V_{\min}0 < 0$, that gives

$$w = \frac{\frac{1}{2}\,\dot{\phi}^2 - V}{\frac{1}{2}\,\dot{\phi}^2 + V} \mapsto \frac{\frac{1}{2}\,\dot{\phi}^2 - V_{\min}}{\frac{1}{2}\,\dot{\phi}^2 + V_{\min}} > 1.$$

Therefore, the kinetic energy and Hubble rate will rapidly fall down during short increment of e-folding, while the state parameter will follow $w \gg 1$, until the potential will become to grow. Then, we suggest the existence of second local minimum of V at $V(0) = 0$ giving the flat vacuum. The decay of flat vacuum to the Anti-de Sitter point $V_{\min} < 0$ is preserved by the gravitational effects [15, 16], so that the flat vacuum can be stable. The barrier between two minima of potential suggests the presence of negative second derivative of potential with respect to the field, that can switch on the preheating mechanism at the scale of potential barrier denoted by $\mu^4_{\text{reh.}}$, which could be much less than the energy density at the end of inflation. Thus, in Fig. 1 the region between the dotted and long-dashed lines can be actual in the extended version of quartic inflation at $|V_{\min}| \ll m^4_{\text{Pl}}$. Though, we cannot point to any verified realistic model of such the scenario to the moment.

The observational data deal with the wave number k about two orders of magnitude less than k_0, that diminish the corresponding number of e-folding by the value about $\delta N \simeq 4 - 5$.

2. Comparing with data

For the flat Universe with cosmological constant, recent WMAP data [14] at $k = 0.002$ Mpc^{-1} gave quite precise values of

$$n_{\text{S}} = 0.951^{+0.015}_{-0.019}, \qquad \mathcal{P}_{\text{S}} = 2.36^{+0.12}_{-0.16} \cdot 10^{-9}, \tag{68}$$

while the slope of n_{S} is not so restrictive

$$\alpha = -0.102^{+0.050}_{-0.043}. \tag{69}$$

Then, we extract the amount of appropriate e-folding

$$N = 61^{+30}_{-15}, \tag{70}$$

that yields the coupling constant equal to

$$\lambda = 1.6^{+2.0}_{-1.2} \cdot 10^{-13}, \tag{71}$$

which follows from Fig. 2.

Note, that the inflation predicts the significant suppression of slope α with respect to the central value in (69). Therefore, an improvement of accuracy for the measurement of slope could serve for discriminating the inflation models.

Further, the WMAP data gave a strong correlations between the spectral index of scalar perturbations n_{S} and fraction of tensor perturbations in the energy density r, that is shown in Fig. 3. The correlations restricts the regions of n_{S} admissible for the quartic-potential inflation, as shown in Fig. 2 by the shaded domain.

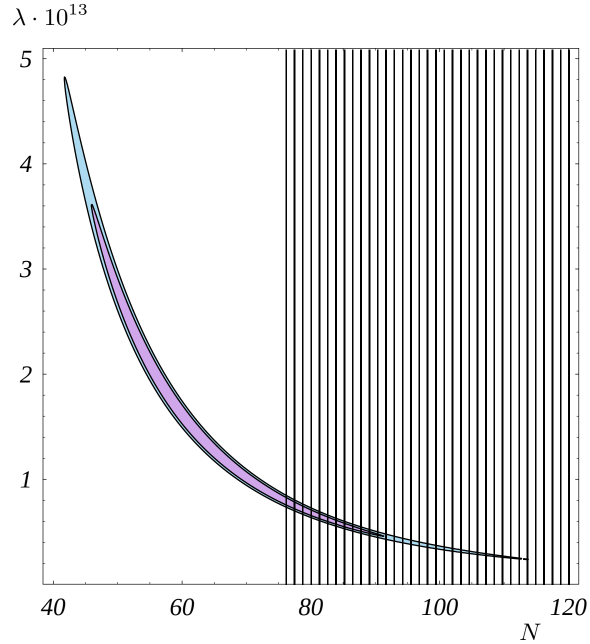

FIG. 2: The experimentally admissible region in the plain of (N, λ) with 1σ and 2σ contours as follows from (68), (69). The shaded region is allowed after taking into account the correlation of spectral index n_{S} with the fraction of tensor perturbations in the energy density r at 2σ level.

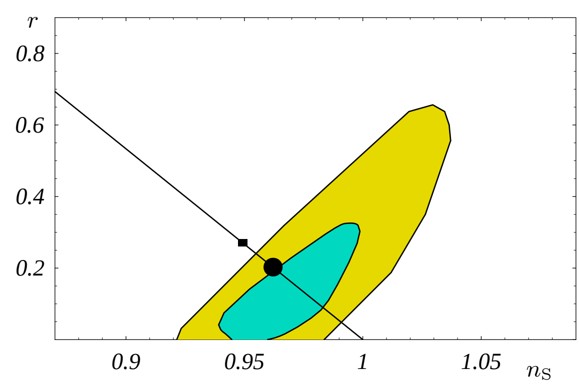

FIG. 3: The WMAP data on the correlations between the spectral index of scalar perturbations n_{S} and fraction of tensor perturbations in the energy density r, as represented by 1σ and 2σ contours. The line gives the predictions of quartic inflation versus large amount of e-folding N. The dot stands at $N = 79$, while the rectangle does at $N = 59$.

Thus, the $\lambda\phi^4$-inflations could be marginally consistent with observations, if one suggest the extended version of preheating at low scales about $\mu_{\text{reh.}} \sim 10^9$ GeV with the preliminary passing the region with negative values

of potential as we can conclude from Figs. 1–3. An example of model for such the potential will be considered elsewhere [17]. However, to the moment there is no a safe realistic model for the preheating scenario consistent with the observational data in the quartic inflation.

IV. CONCLUSION

We have shown that the inflationary dynamics with the quartic potential obeys the parametric quasiattractor governed by the Hubble rate slowly evolving with e-folding of expansion. The condition of attractor stability is preserved by the condition of accelerated expansion.

The quasiattractor allows us to express the inflationary parameters in terms of coupling constant λ and amount of e-folding N in consistence with the slow-roll approximation. Sub-leading terms to the approximation are also on hands.

For the case of quartic potential, we have re-analyzed the possible maximal amount of e-folding N_0 corresponding to the scale of astronomical observations measuring the inhomogeneities generated by the quantum perturbations of inflaton just before the end of inflation. It is spectacular that at the reheating scale $\mu_{\rm reh.} \sim 0.3 \cdot 10^{16}$ GeV, the value of $N_0 \approx 64$ is independent of the particular mechanism of reheating parameterized by the state parameter $w_{\rm reh.}$. At the low-scale reheating the maximal N_0 could be increased due to the essential modification of potential near the origin. For instance, the field could pass the region of negative potential with the further relaxation in the flat minimum at $V = 0$ after overcoming the barrier producing the tachyonic preheating.

Then, the standard quartic inflation with realistic parameters of preheating regime is inconsistent with the observations of matter-density fluctuations, its spectral index of scalar perturbations and fraction of tensor fluctuations. However, there is the marginal case at the coupling constant $\lambda \sim 6 \cdot 10^{-14}$ and amount of e-folding $N \sim 80$ in the modified preheating at low-scales with the negative valley of potential described above, so that the quartic potential of inflaton at high fields with the appropriate modification near the origin is generally still not excluded, but a realistic model with such the scenario is not known.

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